Heisenberg-Limited Uncertainty for Ultrafast Electromagnetic Pulses

The common form of the Heisenberg Uncertainty Principle, $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{4\pi}$, is derived from the commutator relation: $\sigma_f \cdot \sigma_g \geq \frac{1}{2} |\langle f | g \rangle|$. Although time is not an operator, an energy-time relation can be drawn from the above formula:

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{4\pi}$$

and from this, several other relationships,

$$\Delta \nu \cdot \Delta t \geq \frac{1}{4\pi}$$

since frequency is proportional to energy, all energy relations can be derived from this.

recall that $\nu = \frac{c}{\lambda}$ and let $\lambda_1 = \lambda_c + \frac{1}{2} w$ and $\lambda_2 = \lambda_c - \frac{1}{2} w$, where $w$ is the full width, then

$$\Delta \nu = \frac{c \cdot w}{\lambda_c^2 - \frac{w^2}{4}}$$

and

$$\Delta t = \frac{\lambda_c^2 - \frac{w^2}{4}}{4\pi \cdot c \cdot w}$$

**Energy Solution**

$$\Delta \nu \geq \frac{1}{4\pi \cdot \Delta t} \quad \Delta \nu (\Delta t) := \frac{1}{4\pi \cdot \Delta t}$$

**Wavelength Solution**

$$w = \left\{ \begin{array}{ll} -8 \pi \cdot c \cdot \Delta t + 2 \sqrt{16 \pi^2 \cdot c^2 \cdot \Delta t^2 + \lambda_c^2} \\ -8 \pi \cdot c \cdot \Delta t - 2 \sqrt{16 \pi^2 \cdot c^2 \cdot \Delta t^2 + \lambda_c^2} \end{array} \right.$$  \hspace{1cm} \text{formula for width}

$$w(\Delta t, \lambda_c) := -8 \pi \cdot c \cdot \Delta t + 2 \sqrt{16 \pi^2 \cdot c^2 \cdot \Delta t^2 + \lambda_c^2}$$
t := offset ← 13
   for  i ∈ 0 .. 5
   for  j ∈ 10 .. 100
       \( T_{100 \cdot i+j-10 \cdot (i+1)} \leftarrow \frac{j}{10} \cdot 10^{-(i+\text{offset})} \)
   sort(T)

calculates 90 data points for each order of magnitude of time

\[
\begin{align*}
t &:= \text{submatrix}(t, 0, \text{rows}(t) - 1, 0, 0) \\
t &:= \text{t sec} \\
i &:= 0 .. \text{rows}(t) - 1
\end{align*}
\]
Heisenberg Uncertainty Broadening

Frequency Uncertainty /Hz

Pulse Duration /sec

Wavenumber Uncertainty /cm⁻¹

Pulse Duration /sec

Heisenberg Uncertainty Broadening

Frequency Uncertainty /Hz

Wavenumber Uncertainty /cm⁻¹

Heisenberg Uncertainty Broadening

Frequency Uncertainty /Hz

Wavenumber Uncertainty /cm⁻¹
Heisenberg Uncertainty Broadening

\[ \lambda \]

Wavelength Uncertainty (in nm)

Pulse Duration (in seconds)

\[ \lambda_c \]

2000 nm

500 nm

300 nm

200 nm

100 nm

50 nm